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A new class of non-Hermitian Hamiltonians with real spectra

T V Fityo

Ivan Franko National University of Lviv, Chair of Theoretical Physics, 12 Drahomanov Street, Lviv UA-79005, Ukraine

E-mail: fityo@ktf.franko.lviv.ua

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Abstract
We construct a new class of non-Hermitian Hamiltonians with real spectra. The Hamiltonians possess one explicitly known eigenfunction.

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1. Introduction

Recently, non-Hermitian Hamiltonians have attracted much attention. Such Hamiltonians are used in optics [1, 2], in field theory [3] and in other branches of theoretical physics.

Among the non-Hermitian Hamiltonians, much attention has been devoted to the investigation of properties of the so-called PT symmetric Hamiltonians [4–13]. A Hamiltonian is said to be PT symmetric if $PT H = H PT$, where P is the parity operator, i.e. $Pf(x) = f(-x)$, and T is the complex conjugation operator. The main reason for this interest was an assumption that their spectra were entirely real as long as the PT symmetry was not spontaneously broken.

There are several ways to build a non-Hermitian Hamiltonian with a real spectrum. For this purpose, supersymmetric quantum mechanics has been used [14]. In the case of polynomial complex potentials the use of some spectral equivalences has been proposed [15, 16].

Recently, Mostafazadeh has generalized PT symmetry by pseudo-Hermiticity [17]. The idea of pseudo-Hermiticity was introduced by Pauli [18] (see also [19] and references therein). A Hamiltonian H is said to be η-pseudo-Hermitian if

$$H^{+} = \eta H \eta^{-1} \tag{1}$$

where $^{+}$ denotes the adjoint operation. In [17], a new class of non-Hermitian Hamiltonians with real spectra has been proposed, which are obtained using pseudo-supersymmetry.

Mostafazadeh [20] has also shown that the operator H with a complete set of biorthonormal eigenvectors has a real spectrum if and only if there exists a linear invertible operator O such that H is η-pseudo-Hermitian, where $\eta = O^{+}O$.

In this paper we construct a new class of pseudo-Hermitian operators with real spectra using O as a first-order differential operator.

2. Pseudo-Hermiticity

Let us suppose that a non-Hermitian Hamiltonian H is η-pseudo-Hermitian:

$$\eta H = H^{+}\eta. \tag{2}$$

Here, we choose another form of pseudo-Hermiticity to avoid a necessity of η invertibility (the form (2) is mentioned in [17]).

Let us choose an operator η to be a Hermitian operator. Then ηH is also a Hermitian operator: $(\eta H)^{+} = H^{+}\eta^{+} = H^{+}\eta = \eta H$. Consider an eigenfunction ψ and the corresponding eigenvalue E of H. Then, because of the Hermiticity of ηH as well as of η,

$$\int \psi^{*}\eta H\psi \,\mathrm{d}x = E\int \psi^{*}\eta\psi \,\mathrm{d}x \tag{3}$$

both integrals are real and except for the case

$$\int \psi^{*}\eta\psi \,\mathrm{d}x = 0 \tag{4}$$

the eigenvalue E is also real. In contrast, if $\int \psi^{*}\eta\psi \,\mathrm{d}x = 0$ then the left integral of (3) also has to be zero. In this case, E can be either a real or a complex number.

For a general form of η it is difficult to find out if there exist such eigenfunctions which satisfy (4). To simplify the study of the case of $\int \psi^{*}\eta\psi \,\mathrm{d}x = 0$ we concretize the form of η to be

$$\eta = O^{+}O. \tag{5}$$

For this case, the integral $\int \psi^{*}O^{+}O\psi \,\mathrm{d}x = \int |O\psi|^{2}\,\mathrm{d}x$ is greater than zero except for the case of ψ belonging to the kernel of O. So we have to solve

$$O\phi = 0 \tag{6}$$

and verify if solutions of this equation are the eigenfunctions of H.

In the following section we build such a pair of Schrödinger Hamiltonian

$$H = -\frac{\mathrm{d}^{2}}{\mathrm{d}x^{2}} + V(x) \tag{7}$$

and $O^{+}O$ that satisfies the condition (2).

3. *O* as the first-order differential operator

We choose O in the following form:

$$O = \frac{\mathrm{d}}{\mathrm{d}x} + f(x) + \mathrm{i}g(x) \tag{8}$$

where f and g are regular, real-valued functions. Then

$$O^{+} = -\frac{\mathrm{d}}{\mathrm{d}x} + f(x) - \mathrm{i}g(x). \tag{9}$$

Substituting equations (7)–(9) into equation (2) and collecting terms with the $\frac{\mathrm{d}^{2}}{\mathrm{d}x^{2}}$ operator, we obtain

$$\mathrm{Im}\, V = -2g'. \tag{10}$$

The terms without differential operators lead to $4g'(f'+f^2)+2g(f'+f^2)'=g'''$. Multiplying this equation by g and integrating it, we obtain

$$f^2 - f' = \frac{2gg'' - g'^2 + \alpha}{4g^2} \tag{11}$$

where α is a real constant of integration.

The terms with $\frac{\mathrm{d}}{\mathrm{d}x}$ give $2\mathrm{Re}\,V' = 2(f^2 - f' - g^2)'$. Integrating it and substituting equation (11), we can rewrite the real part of the potential as

$$\mathrm{Re}\,V = f^2 - f' - g^2 + \beta = \frac{2gg'' - g'^2 + \alpha}{4g^2} - g^2 + \beta \tag{12}$$

where β is a real constant of integration. In equations (10)–(12) g plays the role of a generating function. In order to obtain a PT symmetric Hamiltonian, the generating function g must be an even function, i.e. $g(x) = g(-x)$.

It should be noted that the choice of O in the form of equation (8) leads to $\eta = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2\mathrm{i}g\frac{\mathrm{d}}{\mathrm{d}x} + f^2 - f' + g^2 - \mathrm{i}g'$ and η plays the role of a second-order Darboux operator. It intertwines H and H^+ which are superpartners of the second-order supersymmetry [21]. Formulae (10)–(12) are similar to the corresponding results of [22].

The next step is to check whether the solution of (6) is an eigenfunction of H. In terms of f and g we can express this solution as

$$\phi = \mathrm{e}^{-\int (f+\mathrm{i}g)\,\mathrm{d}x}. \tag{13}$$

Considering ϕ as an eigenfunction of equation (7) and using equations (10) and (12), we obtain

$$-\mathrm{i}(g' + 2fg) + \beta = E$$

where $E = E_r + \mathrm{i}E_i$ is the complex eigenvalue of H ($H\phi = E\phi$). We see that $\beta = E_r$. Then

$$f = -\frac{E_i + g'}{2g}. \tag{14}$$

Now, from equations (14) and (11) we have two different relations between f and g. To compare them, we substitute f from equation (14) into equation (11) and, after some simplification, we obtain $E_i^2 = \alpha$. So we can state that ϕ can be an eigenfunction of equation (7) only if $\alpha \geqslant 0$. Note that equation (14), for the case $E_i^2 = \alpha$, is the solution of equation (11).

So, by choosing any g and $\alpha < 0$, we can be sure that the spectrum of the corresponding Hamiltonian is entirely real, but we are not sure that it is not empty. By choosing a suitable g for $\alpha = 0$, one can construct the Hamiltonian with a real spectrum and this also possesses one explicitly known eigenfunction. Choosing g and $\alpha > 0$, we have to check if the corresponding ϕ does not belong to L_2 space to obtain a Hamiltonian with a real spectrum.

In the following section, we illustrate these results.

4. Examples

To construct Hamiltonians, we use formulae (10) and (12) to represent the imaginary and real parts of the potential, as well as using equation (14) to express f. There are two ways to obtain a regular expression for f. The first is to choose g without changing the sign and any value of E_i or α. This is illustrated by example 1. The second way is to choose g as a function with a simple zero. In this case, we have to fix the value of the E_i to avoid singularity. This way is illustrated by examples 2 and 3.

Example 1

By choosing the generating function g as the even function

$$g = \mathrm{e}^{-x^2}$$

we obtain the PT symmetric Hamiltonian

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \frac{\alpha}{4}\,\mathrm{e}^{2x^2} - \mathrm{e}^{-2x^2} - 4\mathrm{i}x\,\mathrm{e}^{-x^2} + \beta - 1 \tag{15}$$

which possesses a real spectrum for $\alpha < 0$. For $\alpha = 0$ we know one eigenfunction $\psi_{E=\beta} = \exp\left(-\frac{x^2}{2} - \mathrm{i}\int \mathrm{e}^{-x^2}\,\mathrm{d}x\right)$, and for $\alpha = E_i^2 > 0$ the eigenfunction $\psi_{E=\beta+\mathrm{i}E_i} = \exp\left(-\frac{x^2}{2} + \frac{E_i}{2}\int \mathrm{e}^{x^2}\,\mathrm{d}x - \mathrm{i}\int \mathrm{e}^{-x^2}\,\mathrm{d}x\right)$ does not belong to L_2 space. So we can state that the spectrum of equation (15) is entirely real for any value of the α parameter.

Example 2

Let us choose the generating function g in the form

$$g = \sinh(x).$$

Then, to obtain regular $f = -\frac{E_i + \cosh(x)}{2\sinh(x)}$ one must set $E_i = -1$ and then $f = -\frac{1}{2}\tanh\frac{1}{2}x$. Then $\phi = \cosh\left(\frac{1}{2}x\right)\mathrm{e}^{-\mathrm{i}\cosh(x)}$ does not belong to L_2. So, the spectrum of

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - 2\mathrm{i}\cosh(x) - \sinh^2(x)$$

is real.

Example 3

Let us choose the generating function g in the form

$$g = \tanh(x).$$

Then, avoiding singularity, we set $E_i = -1$ and obtain

$$H = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{2\mathrm{i} - \frac{1}{4}}{\cosh^2(x)} + \beta - \frac{3}{4} \tag{16}$$

with the eigenfunction $\psi_{E=\beta-\mathrm{i}} = \left(\frac{1}{\sqrt{\cosh(x)}}\right)\mathrm{e}^{-\mathrm{i}\ln(\cosh(x))}$. The spectrum of this Hamiltonian can be found using supersymmetric methods and it is easy to show that this eigenvalue is unique.

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